Can Loss Aversion Explain Ambiguity Aversion?

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September 23, 2025

Abstract

We propose a new model of preferences over uncertain outcomes to explain ambiguity aversion. The model combines the insights of loss aversion (Kahneman and Tversky, 1979) with the two-stage approach of previous models, primarily that of Segal (1987). The model is similar in flavor to the *vector expected utility* model proposed by Siniscalchi (2009), in which acts are evaluated with respect to their expected utility combined with an adjustment function. More generally, Grant and Polak (2013) show that the model of Siniscalchi (2009)—and several other models of ambiguity aversion—are special cases of what they call *mean-dispersion preferences*, in which acts are evaluated with respect to their mean utility accounting for deviations from the mean. The key difference in the model we propose is that acts are evaluated differently depending on whether they are presented in isolation or alongside another act (capturing the insights of Fox and Tversky (1995)), and preferences are centered around a *reference utility* rather than the expected utility. Because of this, the model we propose is not a form of mean-dispersion preferences (Grant and Polak, 2013) or even the more general dispersion aversion model of Chambers et al. (2014). Examples are provided to demonstrate the model's ability to capture observed behavior across several different settings, providing an intuitive explanation for ambiguity aversion.

1 Introduction

A continuing open question in economics is why individuals display aversion to ambiguity. This phenomenon was first introduced by Ellsberg (1961) in thought experiments which have since become known as *Ellsberg Urns*. In this paper, we propose a new model of decision making that incorporates loss aversion to explain the existence of ambiguity aversion. To motivate the model, we focus our discussion on the two-color urn thought experiment first introduced by Ellsberg.

The two-color urn experiment can be set up as follows: there are two urns, each containing 100 balls. In the first urn, there are 50 red and 50 black balls. In the second urn, the mixture of red and black balls is ambiguous. The decision maker (DM) can choose to place a bet on the first urn or the ambiguous urn. Whichever urn the DM chooses, they can select to place their bet on either a red ball or a black ball. One ball is then randomly drawn from the selected urn, and if the color matches the DM's bet, the DM is awarded some payoff (say \$100). If the color does not match the DM's bet, the DM receives nothing (that is, \$0). Regardless of which urn the DM chooses to place their bet on, their odds of winning are 1/2. However, despite this fact, in practice it has been consistently observed that DMs prefer the first urn over the second, ambiguous urn. That is, individuals display ambiguity aversion.

Standard expected utility theory predicts that bets over the two urns should be equivalent, and therefore cannot explain the observed behavior. In response, many alternative theories of ambiguity have been developed in an attempt to explain what has been coined *Ellsberg preferences.* Among these models, at least three distinct approaches have been used. The first approach, the *Maxmin Expected Utility* model developed by Gilboa and Schmeidler (1989), assumes DMs cannot form a prior for the ambiguous urn and instead consider a set of possible priors, evaluating the bet based on the minimum expected utility obtained over all priors in the set. In our current example, if the DM considers all distributions of red and black balls to be possible, their expected utility will be minimized when the ambiguous urn happens to contain 0 red balls and 100 black balls (assuming the DM bets on red). As such, under the *Maxmin Expected Utility* model the DM will prefer the first urn (with a guaranteed 50% chance of winning) to the ambiguous urn. A second approach has been to allow DMs to have beliefs about the colors of the balls in the ambiguous urn that are represented by non-additive probability measures (Schmeidler, 1989). In our example, the DM may have beliefs represented by a *capacity* $\nu(\cdot)$, where

$$\nu(s_r, s_b) = 1, \text{ and } \nu(s_r) = \nu(s_b) = \varepsilon, \tag{1}$$

for some $\varepsilon \in [0, 1/2)$. Expected utility is then evaluated using the *Choquet Integral.* A third approach has been to model ambiguity as part of a "two-stage" model, separating the first-stage horse lottery from the roulette lottery realized in the second stage (Segal, 1987; Klibanoff et al., 2005). It is within this approach that we develop our model.

Unlike previous models, we develop a new model which incorporates the insights of loss aversion (Kahneman and Tversky, 1979) into individuals' evaluation of the first-stage uncertainty. That is, when comparing the first urn with the ambiguous urn, the DM forms priors about the possible distributions of colors present in the ambiguous urn, and they evaluate these distributions relative to a *reference distribution*. While the reference distribution used could potentially vary across individuals and settings, in our current example we have a straightforward choice: the DM should compare all possible distributions to the 50-50 distribution they receive by choosing the first urn. Any distribution that provides worse odds than the 50-50 bet is considered to be a loss, and any distribution providing better odds is considered a gain. Because individuals are assumed to be loss averse, they will avoid the ambiguous urn because the potential losses outweigh the potential gains.

This model has a significant advantage over previous two-stage models that model ambiguity aversion as concavity of preferences over first-stage outcomes, such as the *Smooth Ambiguity Preferences Model* of Klibanoff et al. (2005). These models are unable to explain why individuals display ambiguity aversion in some cases and ambiguity preference in others, depending on the relative probabilities and whether payoffs are framed in terms of gains or losses. For a given individual, such models must either always predict ambiguity aversion or always predict ambiguity preference. By allowing for loss aversion over first-stage uncertainty, the model we propose can explain such reversals, just as prospect theory has been able to do for risk aversion.

Finally, this new model is also able to provide an intuitive explanation for the recent finding that individuals prefer larger ambiguous urns over smaller ones (Filiz-Ozbay et al., 2022). As the size of the ambiguous urn increases, the set of possible distributions becomes less extreme.[1] As such, a DM who displays loss aversion over first-stage uncertainty will prefer the larger urn.[2]

In what follows, we introduce the model in the context of the two-urn problem. We then provide a characterization of the model without enforcing any structure on decision-makers' beliefs over the distribution of second-stage lotteries, and then use this characterization to derive key results addressing notable findings in the experimental literature on ambiguity aversion. Following this, we introduce additional axioms that provide structure on decision-makers' beliefs over the distribution of second-stage lotteries and, using this additional structure, derive additional results demonstrating the model's ability to capture observed behavior. We then provide numerical examples.

2 Model Setup

We model the ambiguity as a two-stage process. The general setup follows that of Segal (1987). In the first stage, a probability distribution μ over the state space S is realized, being randomly drawn with probability measure M over $\Delta(S)$. In the first urn, the first stage is degenerate, so that the probability measure M collapses to $\mu = (50r, 50b)$ with probability 1. For the second urn, all

[1] An urn with only two balls has only three possible distributions: $(2r, 0b)$, $(1r, 1b)$, or $(0r, 2b)$. For an urn with $N = 100$ balls, there are $N + 1 = 101$ total possible distributions, but the extremes remain the same as for the $N = 2$ urn. That is, there are now many more (98 more, to be exact) possible distributions which are not as extreme as those available in the smaller urn.

[2] This is assuming the DM has beliefs about the measure $\mu(\cdot)$ which are centered around the reference distribution $(50r, 50b)$ and their uncertainty over the measure $\mu(\cdot)$, represented by the measure $M(\cdot)$ over the space $\Delta(S)$, assigns nonzero probability to all $\mu \in \Delta(S)$.

combinations of red and black balls are possible, and the DM must form beliefs about the probability measure M over all distributions $\mu \in \Delta(S)$. In the first stage, a probability distribution $\mu \in \Delta(S)$ is realized. In the second stage, a state s is then drawn based on the probability distribution μ. As done in previous models, we assume the DM views the two stages as separate and distinct. We also assume that all measures have finite support.

The DM views the basic act $f(\cdot) = (...; x_j, E_j; ...)$ as occurring within the two-stage prospect $(...; (...; x_j, E_j; ...), \mu_k; ...)$, where μ_k is the probability distribution over states that is realized in the first stage, and E_j is the event realized under distribution μ_k in the second stage. Given an act $f(\cdot) = (...; x_j, E_j; ...)$, each probability distribution $\mu \in \Delta(S)$ induces a simple lottery of the form $(...; x_j, \mu(E_j); ...)$. Thus, each basic act $f(\cdot)$ can be represented as a two-stage objective lottery of the form $(...; (...; x_j, \mu_k(E_j); ...), M(\mu_k); ...)$, where $\mu_k(E_j)$ is the probability of event E_j occurring given the probability distribution μ_k, and $M(\mu_k)$ is the individual's *personalistic* or *subjective probability* of distribution μ_k being realized in the first stage.

Following Segal (1987), we assume the individual has a preference function $V(\cdot)$ defined over single-stage lotteries, and they are able to use this preference function to determine the *certainty equivalent* of each single-stage lottery. Thus, the DM uses $V(\cdot)$ to determine the certainty equivalent of each single-stage lottery induced by each probability distribution $\mu \in \Delta(S)$. That is, for each probability distribution $\mu \in \Delta(S)$, the DM calculates the certainty equivalent $CE(f, \mu)$ for the lottery induced by μ, such that

$$V(CE(f, \mu), 1) = V(...; x_j, \mu(E_j); ...).$$

The two-stage lottery faced by the DM can now be rewritten as

$$(...; (...; x_j, \mu_k(E_j); ...), M(\mu_k); ...) = (...; CE(f, \mu_k), M(\mu_k); ...). \tag{2}$$

If the individual evaluates this lottery again using the function $V(\cdot)$, this yields the preference function

$$W(f(\cdot)) \equiv V(...; CE(f, \mu_k), M(\mu_k); ...).$$

In this case, if the preference function $V(\cdot)$ over objective lotteries is expected utility, the final preference function $W(f(\cdot))$ just becomes subjective expected utility, which is unable to explain Ellsberg preferences. Thus, rather than using expected utility, Segal (1987) uses the rank-dependent expected utility form of Quiggin (1982) for $V(\cdot)$.

From here we take a different approach. We use the expected utility form for $V(\cdot)$,[3] and we assume that individuals form additive probability measures M over the possible probability distributions $\mu \in \Delta(S)$. However, we assume that individuals do not have the same preference function for subjective (first-stage) and objective (second-stage) lotteries. In particular, subjects have the

[3] That is, for a lottery $\mathbf{P} = (...; \alpha_i, p_i; ...)$, $V(\mathbf{P}) = \sum_i U(\alpha_i) \cdot p_i$.

expected utility preference function $V(\cdot)$ for second-stage lotteries, but have a different preference function $\psi(\cdot)$ for first-stage lotteries. The preference function $\psi(\cdot)$ over first-stage lotteries incorporates loss aversion, and is evaluated with respect to some reference lottery $\mathbf{P}_r$. For a lottery $\mathbf{P} = (...; \alpha_i, p_i; ...)$ with $\alpha_1 > \cdots > \alpha_m > \cdots > \alpha_n$, where $\alpha_m = \inf\{\alpha : \ \alpha \geq r, \ r \equiv U^{-1}(V(\mathbf{P}_r))\}$, the preference function takes the form

$$\psi(\mathbf{P}, r) = U(r) + \sum_{i=1}^{m} \upsilon_+(\alpha_i - r) \cdot p_i + \sum_{j=m+1}^{n} \upsilon_-(\alpha_j - r) \cdot p_j, \tag{3}$$

where

$$U(x) = \begin{cases} \upsilon_+(x) & \text{for } x \geq 0 \\ \upsilon_-(x) & \text{for } x < 0 \end{cases} \tag{4}$$

and, consistent with loss aversion, $\upsilon_+(x) > 0$, $\upsilon_-(x) < 0$, $\upsilon'_+(x) < \upsilon'_-(-x)$, $\upsilon''_+(x) \leq 0$, and $\upsilon''_-(-x) \geq 0$ for all $x > 0$; and $\upsilon_+(0) = \upsilon_-(0) = 0$.

In general, an exact formula for identifying the individual's reference point may be difficult to pin down. However, with respect to the motivating example of the two-urns problem, there exists a very straightforward reference point: the probability distribution $\mu = (50r, 50b)$ available to the DM with certainty via the first urn. Denoting this probability distribution as μ_r, the individual compares the certainty equivalent of the lottery induced by each $\mu \in \Delta(S)$ with the certainty equivalent of the lottery induced by μ_r. Labeling the induced lotteries $\mathbf{P}_k = (...; x_j, \mu_k(E_j); ...)$ so that $V(\mathbf{P}_1) \geq \cdots \geq V(\mathbf{P}_m) \geq \cdots \geq V(\mathbf{P}_n)$, where $V(\mathbf{P}_m) = \inf\{V(\mathbf{P}_i) : \ U^{-1}(V(\mathbf{P}_i)) \geq U^{-1}(V(\mathbf{P}_r))\}$, the individual's preference function for the second urn can be written as

$$W(f(\cdot)) = V(\mathbf{P}_r) + \sum_{i=1}^{m} \upsilon_+\big(CE(f, \mu_i) - CE(f, \mu_r)\big) \cdot M(\mu_i) + \sum_{j=m+1}^{n} \upsilon_-\big(CE(f, \mu_j) - CE(f, \mu_r)\big) \cdot M(\mu_j). \tag{5}$$

The first term comes from observing that the certainty equivalent of a lottery $\mathbf{P}$ is given by $U^{-1}(V(\mathbf{P}))$. Without any ambiguity (i.e., without any first-stage uncertainty over the distribution μ), the above simplifies to become $W(f(\cdot)) = V(\mathbf{P}_r)$. Therefore, one restriction we will need to enforce on the formulation of the reference point will be that, for any degenerate first-stage lottery, the reference point used will always be equal to the certainty equivalent of the second-stage lottery which is realized with certainty. That is, if an individual faces a single-stage lottery $\mathbf{P}^*$, their valuation of the degenerate two-stage lottery $\{\mathbf{P}^*, 1\}$ must evaluate to $W(f(\cdot)) = V(\mathbf{P}^*)$.

The general idea of this model is that, when comparing the ambiguous urn to the unambiguous urn, the DM considers all of the possible distributions $\mu \in \Delta(S)$ that could occur in the ambiguous urn, and they weight the 'worse' distributions more than the 'better' distributions. The worse distributions are those that provide a lower certainty equivalent. In the two urn example, the worse distributions are simply those that provide a lower probability of winning. For example, if one is planning to bet on a red ball being drawn, a worse distribution would be one such as $(40r, 60b)$, $(30r, 70b)$, etc. Because all of the possible distributions are centered around the $(50r, 50b)$ distribution of the unambiguous urn, there are an equal number of worse and better distributions possible in the ambiguous urn. As long as the individual forms $M(\cdot)$ to be symmetric around $\mu = (50r, 50b)$, each 'worse' distribution will have a mirror-image 'better' distribution that occurs with equal probability. However, because individuals are loss-averse, the worse distributions are more influential than the better distributions, causing the ambiguous urn to be considered worse than the unambiguous urn.

In the following section, we formally characterize the model.

3 Characterization

3.1 Primitives and Environment

Let S be a finite state space and $X \subseteq \mathbb{R}$ a (bounded, closed) set of monetary outcomes.[4] Write $\Delta(S)$ for the set of probability measures on S. An *act* is a function $f : S \to X$. A (second-stage) *objective lottery* is a simple lottery $P = (x_i, p_i)_{i=1}^n$ on X.

Two-stage ambiguity. Following the two-stage approach, a first-stage draw selects a distribution $\mu \in \Delta(S)$ according to a (subjective) probability measure M on $\Delta(S)$ with finite support; then a state s is drawn from μ and $f(s)$ is paid. We denote the (compound) *ambiguous act* by (f, M). For $\mu \in \Delta(S)$, let $P_{f,\mu}$ denote the objective lottery induced by f under μ: $P_{f,\mu} = (f(s), \mu(s))_{s \in S}$.

Second-stage EU and certainty equivalents. Preferences over objective lotteries satisfy vNM axioms (below), so there exists a strictly increasing, continuous Bernoulli utility $u : X \to \mathbb{R}$ such that

$$V(P) = \sum_i u(x_i) p_i \quad \text{for any objective lottery } P = (x_i, p_i)_i.$$

The certainty equivalent (CE) of P is $\text{CE}(P) = u^{-1}(V(P))$. We write $\text{CE}(f, \mu) := \text{CE}(P_{f,\mu})$.

Comparison contexts and references. Choices are *contextual* in the sense that the decision maker (DM) evaluates a target act relative to a *comparison act* that pins down a reference point.[5] Formally, a *context* is an ordered pair $c = (\mathcal{A}; g)$, where $\mathcal{A}$ is a feasible set containing the target act and $g \in \mathcal{A}$ is the designated comparison act.[6] A *reference selection operator* R maps g to an

[4] The monetary domain simplifies exposition; standard extensions to general outcome spaces with a Bernoulli utility follow.

[5] This is where the model departs from mean/dispersion preferences: the reference is context-generated, not the mean of the target object.

[6] In binary choice experiments, g can be taken as "the other option." In unilateral evaluation, g can be a status quo.

objective reference lottery $P^r = R(g)$ that will anchor the first-stage evaluation of the target.

The DM has a context-dependent preference relation $\succeq_c$ over acts, read "weakly preferred in context c".

3.2 Reference Rules (Selection Axioms)

The following rules formalize the reference valuation. These rules determine which act in $\mathcal{A}$ will serve as the comparison act, and they also determine how the comparison act will be valued. When presented with multiple acts, the DM is assumed to select the least ambiguous option to serve as the reference, where "least ambiguous" is as defined below. If multiple options are tied for the least ambiguous, the DM is assumed to select the act with the highest certainty equivalent. In the two-urn problem where a DM is presented with a choice between an unambiguous urn and an ambiguous urn, it is assumed the DM will select the unambiguous urn as the comparison act.

Axiom 1 (Selection of the comparison act (least-ambiguity rule))**.** Let $\mathcal{A}$ be a finite, nonempty menu of acts. For each $a \in \mathcal{A}$, let f_a be its outcome map on a (finite) state space S_a. Assume Axiom 5 (vNM on risk) and Axiom 3 (label invariance).

Pre-reference baseline for selection. For each $a \in \mathcal{A}$ there exists a σ-finite Borel measure Π_a^0 on $\Delta(S_a)$ such that $\mu \mapsto \mathrm{CE}(f_a, \mu)$ is Borel and integrable under Π_a^0. Denote the *baseline CE* random variable

$$C_a(\mu) := \mathrm{CE}(f_a, \mu), \qquad \mu \sim \Pi_a^0,$$

and its mean $\bar{c}(a) := \mathbb{E}_{\Pi_a^0}[C_a]$. (These Π_a^0 are used *only* to select the comparison; first-stage beliefs for valuation are specified elsewhere.)

Ambiguity (dispersion) index. Fix once and for all a strictly increasing, strictly convex function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$ (e.g., $\rho(t) = t^2$ or $\rho(t) = t$). Define the ambiguity index of a by

$$\mathrm{Amb}(a) \ := \ \mathbb{E}_{\Pi_a^0}\Big[\rho\big(|C_a - \bar{c}(a)|\big)\Big].$$

Intuitively, $\mathrm{Amb}(a)$ measures the *spread* of plausible certainty equivalents for a across models μ.

Selection rule. The comparison act $g \in \mathcal{A}$ is selected by the lexicographic rule

$$g \ \in \ \arg\min_{a \in \mathcal{A}} \Big(\mathrm{Amb}(a), \ -\bar{c}(a)\Big),$$

and, if several acts tie on both coordinates, by a fixed exogenous total order $\preceq$ on labels (choose the $\preceq$-minimal act).

Reference assignment. Given the selected g, the reference lottery is $R(g)$ as specified in Axiom 2 (in particular, if g is an objective lottery P, then $R(g) = P$).

Axiom 2 (Cross-reduction for ambiguous comparisons)**.** If $g = (h, M')$ is ambiguous, then $R(g)$ equals the reduction of g, i.e.,

$$R(g) = \big(u^{-1}(\mathbb{E}_{\mu\sim M'}[V(P_{h,\mu})]), 1\big).$$

Equivalently, $R(g)$ is the degenerate lottery paying $\mathrm{CE}(h, \mu)$ in expectation over M'.

Axiom 3 (Context symmetry and neutrality)**.** R depends on g only via the reduced objective lottery induced by g, and is invariant to outcome/state relabelings that leave that reduced lottery unchanged.

3.3 Axioms on Preferences

Axiom 4 (Weak order)**.** For each context c, $\succeq_c$ is complete and transitive on the feasible set.

Axiom 5 (vNM on objective lotteries)**.** On the subset of objective lotteries, $\succeq_c$ (for any c) satisfies independence and continuity, yielding the u-expected utility representation V above.

Axiom 6 (First-stage separability via CEs)**.** For any context c with reference $P^r = R(g)$ and $r := \mathrm{CE}(P^r)$, the evaluation of a target (f, M) in context c depends on f and M only through the distribution of $\mathrm{CE}(f, \mu)$ under M and the scalar r.

Axiom 7 (Additive aggregation over first-stage uncertainty)**.** Fix c and r. If M has finite support $\{\mu_k\}_{k=1}^K$ with weights $\{M_k\}$, then the value of (f, M) equals a (reference-centered) probability-weighted sum of (possibly different) gain/loss transforms of $\mathrm{CE}(f, \mu_k) - r$.

Axiom 8 (Loss aversion with concave gains and convex losses)**.** There exist continuous, strictly increasing functions $v_+, v_- : [0, \bar{d}] \to \mathbb{R}_+$ with $v_+(0) = v_-(0) = 0$, and a parameter $\lambda > 1$, such that the reference–indexed evaluation map $\phi_r : [-\bar{d}, \bar{d}] \to \mathbb{R}$ is defined by

$$\phi_r(d) = \begin{cases} v_+(d), & d \geq 0, \\ -\lambda\, v_-(-d), & d < 0, \end{cases}$$

and the following properties hold:

(i) **Concave gains.** v_+ is concave on $[0, \bar{d}]$ (diminishing sensitivity for gains).

(ii) **Convex losses for $\boldsymbol{\phi_r}$.** v_- is concave on $[0, \bar{d}]$; hence the loss branch $d \mapsto -\lambda\, v_-(-d)$ is convex on $[-\bar{d}, 0]$.[7]

(iii) **Loss-aversion kink at the reference.** The one-sided slope limits at 0 exist and satisfy

$$\alpha_+ \;:=\; \lim_{x\downarrow 0} \frac{v_+(x)}{x} \in (0, \infty), \qquad \alpha_- \;:=\; \lim_{x\downarrow 0} \frac{v_-(x)}{x} \in (0, \infty),$$

[7]Since v_- is concave, $-v_-$ is convex; composition with the affine map $d \mapsto -d$ preserves convexity.

with the *loss-aversion* inequality

$$\phi'_-(0) \;=\; \lambda\,\alpha_- \;>\; \alpha_+ \;=\; \phi'_+(0).$$

Equivalently, for sufficiently small $\varepsilon > 0$, $|\phi_r(-\varepsilon)| > \phi_r(\varepsilon)$.

(iv) **Monotonicity.** ϕ_r is strictly increasing on $[-\bar{d}, \bar{d}]$.

Axiom 9 (Outcome/utility normalization)**.** u is unique up to positive affine transformations; v_+ and v_- are unique up to a common positive scaling that is absorbed by u's affine normalization.

Axiom 10 (Continuity in M)**.** $\succeq_c$ is continuous in the weak topology induced by convergence of the distributions of $\mathrm{CE}(f, \mu)$ under M.

3.4 Representation

Define the piecewise transform

$$\phi_r(d) := \begin{cases} v_+(d) & \text{if } d \geq 0, \\ -\,\lambda\, v_-(-d) & \text{if } d < 0. \end{cases}$$

Let $r = \mathrm{CE}\big(R(g)\big)$. For (f, M) with finite support $\{\mu_k\}_k$, write $d_k := \mathrm{CE}(f, \mu_k) - r$.

Theorem 1 (Loss-Reference Ambiguity Representation)**.** *Suppose Axioms 1–10 hold. Then for each context $c = (\mathcal{A}; g)$ there exist:*

- *a vNM utility u representing $\succeq_c$ on objective lotteries;*
- *loss-aversion primitives (λ, v_+, v_-) as in Axiom 8;*
- *and an additive first-stage belief M on $\Delta(S)$ (with finite support on considered menus);*[8]

such that $(f, M) \succeq_c (f', M')$ iff

$$W_c(f, M) := u(r) \;+\; \mathbb{E}_{\mu\sim M}\big[\,\phi_r\big(\mathrm{CE}(f, \mu) - r\big)\big] \;\;\geq\;\; u(r) \;+\; \mathbb{E}_{\nu\sim M'}\big[\,\phi_r\big(\mathrm{CE}(f', \nu) - r\big)\big] =: W_c(f', M'), \quad (6)$$

where $r = \mathrm{CE}(R(g))$ is the contextually determined reference given by Axioms 1–2. Conversely, any preference that admits a representation of the form (6) *satisfies Axioms 1–10.*

Remark 1 (Relation to existing classes)**.** Representation (6) is *not* a mean/dispersion functional since (i) it centers at the *contextual* reference r, not the mean of the target, and (ii) the transform is *asymmetric* around r due to $\lambda > 1$. Hence it generally falls outside mean-dispersion and dispersion-aversion classes.

[8] Allowing σ-additivity and integrals over $\Delta(S)$ is straightforward under standard measurability/compactness.

3.5 Comparative Statics and Testable Implications

3.5.1 Ellsberg two-urn aversion (comparative ignorance pattern)

Consider comparing an unambiguous act g to an ambiguous (f, M), with $R(g) = g$, $r = \mathrm{CE}(g)$. Assume M is symmetric around the distribution that replicates g (so that the distribution of $\mathrm{CE}(f, \mu) - r$ is symmetric around 0). Then:

Proposition 1 (Ellsberg aversion under symmetry)**.** *Fix a context $c = (\mathcal{A}; g)$ with g* unambiguous *and reference $r = \mathrm{CE}(g)$. Let (f, M) be ambiguous and let $D := \mathrm{CE}(f, \mu) - r$ (with $\mu \sim M$) be* symmetric *about 0: $D \overset{d}{=} -D$. Assume Axiom 8 (concave v_+ and v_-, $\lambda > 1$) and the* global loss-dominance *condition*

$$\frac{v_+(t)}{v_-(t)} \;\le\; \lambda \qquad \textit{for all } t \in (0, \bar{d}]. \tag{7}$$

Then $W_c(f, M) \le W_c(g, \delta) = u(r)$, with strict inequality whenever $\mathrm{P}(|D| > 0) > 0$ and (7) is strict on a set of $|D|$-positive probability. In particular, (7) is satisfied if $v_- = v_+$, in which case

$$W_c(f, M) = u(r) + \mathbb{E}[\phi_r(D)] \;\le\; u(r), \quad \textit{strict if } \mathrm{P}(|D| > 0) > 0.$$

Remark 2 (About the loss-dominance condition)**.** Condition (7) is a *global* version of the standard local loss-aversion kink ($\lambda\alpha_- > \alpha_+$). It holds trivially when $v_- = v_+$ and $\lambda > 1$, and more generally whenever $\lambda \ge \sup_{t \in (0, \bar{d}]} v_+(t)/v_-(t)$, which is natural if λ is elicited as the *upper envelope* of gain-to-loss sensitivity ratios on the relevant domain.

3.5.2 Reversals as success probability changes

Let g deliver high success probability p_h and g' low p_ℓ with $p_h > 1/2 > p_\ell$. Under symmetric M, the mass of CE deviations below r is larger around a high p_h reference; above r around a low p_ℓ reference. Then:

Proposition 2 (Reference shifts generate reversals)**.** *Fix the primitives and axioms of Theorem 1. Consider a binary-outcome environment with $x_L < x_H$ and, for each $p \in [0, 1]$, let g_p be the* unambiguous *act that yields x_H with objective probability p and x_L otherwise. By Axiom 5,*

$$r(p) := \mathrm{CE}(g_p) = u^{-1}\big(p\, u(x_H) + (1 - p)\, u(x_L)\big)$$

is well-defined, continuous, and strictly increasing in p.

Fix an ambiguous *target act (f, M) and write*

$$X := \mathrm{CE}(f, \mu) \in [x_L, x_H] \qquad (\mu \sim M),$$

which is a bounded, non-degenerate random variable unless (f, M) is degenerate. Let $\phi_r(d) = v_+(d)\mathbf{1}\{d \ge 0\} - \lambda v_-(-d)\mathbf{1}\{d < 0\}$ be the loss-reference transform from Axiom 8, and define for *each p the difference value (ambiguous versus unambiguous benchmark at p):*

$$\Delta W(p) \;:=\; W_c\big(f, M\big) - W_c(g_p, \delta) = \mathbb{E}\big[\phi_{r(p)}\big(X - r(p)\big)\big]\,.$$

Then:

(a) $\Delta W : [0,1] \to \mathbb{R}$ *is continuous and* strictly decreasing *in p.*

(b) $\Delta W(0) > 0$ *provided* $\mathrm{P}(X > x_L) > 0$*, and* $\Delta W(1) < 0$ *provided* $\mathrm{P}(X < x_H) > 0$*.*

Consequently, there exists a unique $\bar{p} \in (0,1)$ such that

$$\Delta W(p) \begin{cases} > 0 & \text{for } p < \bar{p} \quad (\text{ambiguity seeking}), \\ = 0 & \text{for } p = \bar{p}, \\ < 0 & \text{for } p > \bar{p} \quad (\text{ambiguity aversion}). \end{cases}$$

In particular, the DM is ambiguity seeking for sufficiently small p and ambiguity averse for sufficiently large p.

3.5.3 Preference for larger ambiguous urns

Let M_N be beliefs over $\Delta(S)$ with urn size N (e.g., uniform over counts or binomial with 1/2). As N increases, M_N *contracts* toward its center (in convex order). Then:

Proposition 3 (Ambiguity-size effect)**.** *Fix a context $c = (\mathcal{A}; g)$ with reference $r = \mathrm{CE}(R(g))$, and an act f. Let $D := \mathrm{CE}(f, \mu) - r$ (with $\mu \sim M$) and $D' := \mathrm{CE}(f, \mu') - r$ (with $\mu' \sim M'$). Assume:*

(a) ***Symmetry at the reference:*** $D \overset{d}{=} -D$ *and* $D' \overset{d}{=} -D'$ *(e.g., by a symmetric baseline and a kernel depending on $|d|$).*

(b) ***Magnitude contraction:*** $|D'| \leq_{\mathrm{FOSD}} |D|$ *on* $[0, \bar{d}]$ *(first-order stochastic dominance of magnitudes).*

(c) ***Concave common sensitivity:*** $v_+ = v_- =: v$ *is continuous, strictly increasing and concave, with* $v(0) = 0$*.*

(d) ***Loss aversion:*** $\lambda > 1$*.*

Then

$$W_c(f, M') \;\geq\; W_c(f, M),$$

with strict inequality if $\mathrm{P}(|D'| < |D|) > 0$ *and v is strictly increasing on a set of positive $|D|$-mass.*

Appendix: Proofs

Proof of Theorem 1 (Loss-Reference Ambiguity Representation)

We provide a complete proof of the representation in Theorem 1 and its converse. Throughout, fix a context $c = (\mathcal{A}; g)$, let $P^r := R(g)$ be the (objective) reference lottery specified by Axioms **??**–2, and write

$$r := \mathrm{CE}(P^r) \in X, \qquad \mathrm{CE}(P) := u^{-1}\left(\sum_i u(x_i)p_i\right)$$

with u given by Axiom 5. For any ambiguous act (f, M), define the (first-stage) random variable

$$D_{f,M} := \mathrm{CE}(f, \mu) - r \qquad (\mu \sim M).$$

Axiom 6 ensures that, in context c, the evaluation of (f, M) depends on (f, M) only through the *law* $\mathsf{Law}(D_{f,M})$ and the scalar r. We denote by $\mathcal{P}_r$ the set of all Borel probability measures on the compact interval

$$I_r \ := \ [\underline{c} - r, \overline{c} - r] \subset \mathbb{R}, \qquad \underline{c} := \min X, \ \overline{c} := \max X,$$

equipped with the topology of weak convergence.

Plan.

1. Reduce preferences on acts to a continuous *affine* functional $F_r : \mathcal{P}_r \to \mathbb{R}$ over the distributions of CE-deviations (Lemmas 1–3).
2. Show that, for *finite-support* laws, F_r is the expectation of a single-index function $\psi_r : I_r \to \mathbb{R}$ (Lemma 5).
3. Extend to all of $\mathcal{P}_r$ by continuity (Lemma 6).
4. Identify the gain/loss branches from Axiom 8 and normalize (Lemma 7).
5. Verify the converse: any preference represented by (6) satisfies Axioms **??**–10.

Step 1: Reduction to distributions of CE-deviations

Lemma 1 (Reduction)**.** *Fix c and r. There exists a well-defined functional F_r on the set*

$$\mathcal{P}_r^{\mathrm{ach}} \ := \ \{\mathsf{Law}(D_{f,M}) \ : \ (f, M) \in \mathcal{A}\} \subseteq \mathcal{P}_r$$

such that, for any $(f, M) \in \mathcal{A}$,

$$W_c(f, M) \ = \ u(r) + F_r\left(\mathsf{Law}(D_{f,M})\right).$$

Moreover, by Axioms 6, 3, the value $F_r(P)$ depends only on $P \in \mathcal{P}_r$.

Proof. Fix c and r. Axioms 5 and 6 imply that the evaluation of (f, M) depends only on the distribution of $D_{f,M}$ and on r. Define F_r on $\mathcal{P}_r^{\text{ach}}$ by

$$F_r(\mathsf{Law}(D_{f,M})) := W_c(f, M) - u(r).$$

If (f, M) and (f', M') induce the same law for D, Axiom 6 implies $W_c(f, M) = W_c(f', M')$, hence F_r is well-defined on equivalence classes (and thus on the image set $\mathcal{P}_r^{\text{ach}}$). Axiom 3 rules out dependence on labels beyond the law of D. Finally, $D_{f,M} \in I_r$ because $\text{CE}(f, \mu) \in [\underline{c}, \overline{c}]$ and $r \in X$. □

Lemma 2 (Richness and implementability)**.** *For any finite-support* $P = \sum_{k=1}^{K} p_k \, \delta_{d_k}$ *on* I_r *there exists a* single *act* $(\tilde{f}, \tilde{M}) \in \mathcal{A}$ *such that* $\mathsf{Law}(D_{\tilde{f},\tilde{M}}) = P$*.*

Proof. Fix $\{(p_k, d_k)\}_{k=1}^{K}$. For each k, pick any objective lottery P_k on X with $\text{CE}(P_k) = r + d_k$ (possible by continuity and monotonicity of u and the richness of simple lotteries). Construct disjoint finite state spaces S_k and acts $f_k : S_k \to X$ together with beliefs $\mu_k \in \Delta(S_k)$ that implement $P_{f_k,\mu_k} = P_k$. Let $\tilde{S} := \bigsqcup_{k=1}^{K} S_k$ and define $\tilde{f}$ by $\tilde{f}|_{S_k} := f_k$. Define $\tilde{M}$ as the probability measure on $\Delta(\tilde{S})$ which selects μ_k with probability p_k (view each μ_k as supported on its own block S_k). Then, for $\mu \sim \tilde{M}$, $\text{CE}(\tilde{f}, \mu) = \text{CE}(f_k, \mu_k) = r + d_k$ with probability p_k, hence $\mathsf{Law}(D_{\tilde{f},\tilde{M}}) = \sum_k p_k \, \delta_{d_k} = P$. □

Lemma 3 (Affinity in first-stage mixtures)**.** *For any* $P, Q \in \mathcal{P}_r^{\text{ach}}$ *and* $\alpha \in [0, 1]$ *with* $\alpha P + (1-\alpha)Q \in \mathcal{P}_r^{\text{ach}}$*, one has*

$$F_r\big(\alpha P + (1 - \alpha)Q\big) \;=\; \alpha F_r(P) + (1 - \alpha)F_r(Q).$$

Proof. By Lemma 2 there exist acts (f_P, M_P) and (f_Q, M_Q) implementing P and Q. Construct an act (f, M) on the disjoint union state space that equals (f_P, M_P) with probability α and equals (f_Q, M_Q) with probability $1 - \alpha$. This can be done by the same block-diagonal construction as in Lemma 2, now with two blocks and weights $\alpha, 1 - \alpha$. Then $\mathsf{Law}(D_{f,M}) = \alpha P + (1 - \alpha)Q$ and, by Axiom 7,

$$\begin{aligned} F_r(\alpha P + (1 - \alpha)Q) &= W_c(f, M) - u(r) \\ &= \alpha\big(W_c(f_P, M_P) - u(r)\big) + (1 - \alpha)\big(W_c(f_Q, M_Q) - u(r)\big) \\ &= \alpha F_r(P) + (1 - \alpha)F_r(Q). \end{aligned}$$

□

Lemma 4 (Continuity)**.** F_r *is continuous on* $\mathcal{P}_r^{\text{ach}}$ *under weak convergence of laws.*

Proof. Let $P_n = \mathsf{Law}(D_{f_n,M_n}) \to P = \mathsf{Law}(D_{f,M})$ weakly, with all laws in $\mathcal{P}_r^{\text{ach}}$. Axiom 10 delivers continuity of preferences in the weak topology induced by the pushforward of M through $\mu \mapsto \text{CE}(f, \mu)$; hence $W_c(f_n, M_n) \to W_c(f, M)$ and thus $F_r(P_n) \to F_r(P)$. □

Step 2: Affine representation for finite supports

Lemma 5 (Finite-support representation)**.** *There exists a function $\psi_r : I_r \to \mathbb{R}$ with $\psi_r(0) = 0$ such that for any $P = \sum_{k=1}^{K} p_k\, \delta_{d_k} \in \mathcal{P}_r^{\text{ach}}$,*

$$F_r(P) = \sum_{k=1}^{K} p_k\, \psi_r(d_k).$$

Moreover, ψ_r is unique, and ψ_r is strictly increasing and continuous.

Proof. Construction and uniqueness. Define $\psi_r(d) := F_r(\delta_d)$ for $d \in I_r$, where δ_d is the unit mass at d. This is well-defined since (by Lemma 2) every δ_d is achievable via a degenerate first-stage belief (and an appropriate f). Further, Axiom **??** and the definition of r imply that when $d = 0$ (i.e., an act reduced to the reference), $W_c = u(r)$, hence $F_r(\delta_0) = 0$, so $\psi_r(0) = 0$.

Let $P = \sum_{k=1}^{K} p_k\, \delta_{d_k}$ be a finite-support law. Using Lemma 2, build an act (f, M) implementing P by a block-diagonal construction where, conditional on the block k, $\mathsf{Law}(D) = \delta_{d_k}$. Axiom 7 implies

$$F_r(P) = F_r\left(\sum_k p_k \delta_{d_k}\right) = \sum_k p_k F_r(\delta_{d_k}) = \sum_k p_k\, \psi_r(d_k).$$

If also $F_r(P) = \sum_k p_k \tilde{\psi}_r(d_k)$ for some $\tilde{\psi}_r$, then by evaluating at singleton supports we get $\psi_r(d) = \tilde{\psi}_r(d)$ for all $d \in I_r$, proving uniqueness.

Strict monotonicity. Suppose $d_1 < d_2$. Consider the laws δ_{d_1} and δ_{d_2}. By Axiom 4 and u strictly increasing, CE $= r + d_2$ strictly dominates CE $= r + d_1$ as an objective certainty; under Axiom 6, this yields a strict preference at the first stage, hence $F_r(\delta_{d_2}) > F_r(\delta_{d_1})$, i.e., $\psi_r(d_2) > \psi_r(d_1)$.

Continuity. Let $d_n \to d$. Then $\delta_{d_n} \Rightarrow \delta_d$ weakly, so by Lemma 4, $F_r(\delta_{d_n}) \to F_r(\delta_d)$, i.e., $\psi_r(d_n) \to \psi_r(d)$. □

Step 3: Extension to general laws on I_r

Lemma 6 (Continuous affine extension)**.** *Let ψ_r be as in Lemma 5. For any $P \in \overline{\mathcal{P}_r^{\text{ach}}} \subseteq \mathcal{P}_r$ (the closure in the weak topology), define*

$$\mathcal{I}_{\psi_r}(P) := \int_{I_r} \psi_r(d)\, P(dd),$$

where the integral is the (unique) limit of expectations along any sequence of finite-support $P_n \Rightarrow P$. Then $\mathcal{I}_{\psi_r}$ is well-defined, continuous and affine, and $F_r(P) = \mathcal{I}_{\psi_r}(P)$ for all P in the closure. In particular, if $\mathcal{P}_r^{\text{ach}}$ is dense in $\mathcal{P}_r$, then $F_r(P) = \int \psi_r\, dP$ for all $P \in \mathcal{P}_r$.

Proof. Since I_r is compact and ψ_r is continuous, ψ_r is bounded. Let P_n be a sequence of finite-support measures converging weakly to P. By the Portmanteau theorem, for any bounded continuous ψ_r, $\int \psi_r\, dP_n \to \int \psi_r\, dP$. On the other hand, for finite-support laws $P_n = \sum_k p_{n,k}\delta_{d_{n,k}}$, Lemma 5 gives $F_r(P_n) = \sum_k p_{n,k}\psi_r(d_{n,k}) = \int \psi_r\, dP_n$. By Lemma 4, $F_r(P_n) \to F_r(P)$. Therefore the limit

$\lim_n \int \psi_r \, dP_n$ exists, equals $F_r(P)$, and is independent of the chosen approximating sequence. This defines $\mathcal{I}_{\psi_r}(P)$ unambiguously, yields $F_r(P) = \mathcal{I}_{\psi_r}(P)$ on the closure, and continuity/affinity follow from limits and the corresponding properties on finite-support laws. □

Remark 3 (Density of achievable laws)**.** In the baseline environment of the paper we restrict to finite-support M "on considered menus," so Lemma 5 already delivers the representation needed for all objects of interest. If needed, density of finite-support laws in $\mathcal{P}_r$ is standard (empirical distributions approximate any probability law on a compact metric space), and the construction in Lemma 2 ensures implementability of any such finite-support law.

Step 4: Identification of gain/loss branches and the kink

Lemma 7 (Branches and loss aversion)**.** *Let $\psi_r : I_r \to \mathbb{R}$ be the single-index function from Lemma 5, defined by $\psi_r(d) := F_r(\delta_d)$. Under Axiom 8 (concave gains, convex losses for ϕ_r), there exist continuous, strictly increasing, concave functions $v_+, v_- : [0, \bar{c} - r] \to \mathbb{R}_+$ with $v_\pm(0) = 0$ and a parameter $\lambda > 1$ such that*

$$\phi_r(d) := \psi_r(d) = \begin{cases} v_+(d), & d \geq 0, \\ -\lambda \, v_-(-d), & d < 0, \end{cases}$$

and:

- *(a)* ***Curvature by branch.*** *The gains branch $d \mapsto v_+(d)$ is concave on $[0, \bar{c} - r]$. The losses branch $d \mapsto -\lambda \, v_-(-d)$ is* convex *on $[\underline{c} - r, 0]$ (since v_- is concave).*
- *(b)* ***Loss-aversion kink.*** *The one-sided slope limits at 0 exist and satisfy*

$$\phi'_{r,-}(0) = \lambda \, \alpha_- \; > \; \alpha_+ = \phi'_{r,+}(0), \quad \textit{where} \quad \alpha_\pm := \lim_{x \downarrow 0} \frac{v_\pm(x)}{x} \in (0, \infty).$$

- *(c)* ***Monotonicity.*** *ϕ_r is strictly increasing on I_r.*

Proof. By construction of ψ_r from singleton laws (Lemma 5), ψ_r records the first-stage evaluation of a sure deviation d. Axiom 8 postulates that, at such sure deviations, the gain side is represented by a concave v_+, and the loss side by $-\lambda v_-(-\cdot)$ with v_- concave, together with the slope ratio $\lambda\alpha_- > \alpha_+$ at 0. This yields the stated piecewise form. Strict increase follows since $v_\pm$ are strictly increasing and $\lambda > 1$ preserves order on the loss side. Convexity of the loss branch is immediate: v_- concave $\Rightarrow$ $-v_-$ convex; composition with $d \mapsto -d$ preserves convexity; multiplying by $\lambda > 0$ preserves convexity. □

Step 5: Conclusion of the "if" part

Combining Lemmas 1, 5, and 6, and then substituting the branch decomposition from Lemma 7, we have, for any $(f, M) \in \mathcal{A}$,

$$W_c(f, M) \;=\; u(r) \;+\; \int_{I_r} \psi_r(d)\,\mathsf{Law}(D_{f,M})(dd) \;=\; u(r) \;+\; \mathbb{E}_{\mu\sim M}[\phi_r(\mathrm{CE}(f,\mu) - r)]\,,$$

where $\phi_r(d) := \psi_r(d) = v_+(d)\mathbf{1}\{d \geq 0\} - \lambda v_-(-d)\mathbf{1}\{d < 0\}$. This is exactly the representation stated in Theorem 1 for the fixed context c.

Step 6: Uniqueness and normalization

Lemma 8 (Uniqueness)**.** *Within a fixed context c and reference r, the function ψ_r (equivalently ϕ_r) is unique. Consequently, (v_+, v_-, λ) are unique up to a* common *positive multiplicative factor absorbed by the affine normalization of u (Axiom 9).*

Proof. If $F_r(P) = \int \psi_r\, dP = \int \tilde{\psi}_r\, dP$ for all finite-support P, then in particular $\psi_r(d) = \tilde{\psi}_r(d)$ for all d by evaluating at $P = \delta_d$. Hence ψ_r is unique. If we scale ψ_r by $a > 0$ and add a constant b, the constant vanishes in differences because $F_r(\delta_0) = 0$ pins $\psi_r(0) = 0$, so $b = 0$; scaling by a can be absorbed into u's (positive) affine normalization since W_c is only determined up to such transformations, proving the claim about (v_+, v_-, λ). □

Step 7: Converse ("only if")

Assume that for each context $c = (\mathcal{A}; g)$ there exist: (i) u that represents preferences over objective lotteries (Axiom 5); (ii) a reference operator consistent with Axioms **??**–3 that yields $r = \mathrm{CE}(R(g))$; and (iii) a function ϕ_r of the piecewise form in Lemma 7 such that

$$W_c(f, M) = u(r) + \mathbb{E}_{\mu\sim M}\big[\phi_r(\mathrm{CE}(f,\mu) - r)\big]$$

represents $\succeq_c$ on $\mathcal{A}$. We verify the axioms:

- *Axiom 4 (Weak order):* W_c is a real-valued functional; completeness and transitivity follow.
- *Axiom 5 (vNM on risk):* If M is degenerate at a fixed μ, then $\mathrm{CE}(f,\mu) = \mathrm{CE}(P)$ for the objective lottery $P = P_{f,\mu}$ and $W_c(f, M) = u(r) + \phi_r(\mathrm{CE}(P) - r)$. Comparisons between two objective lotteries P and Q with the *same* reference r reduce to u-orderings, as ϕ_r increases strictly and $\phi_r(0) = 0$ pins normalization.
- *Axioms **??**–3 (Reference rules and neutrality):* Hold by assumption on R and because W_c depends on (f, M) only through r and the pushforward law of $\mathrm{CE}(f, \cdot)$.
- *Axiom 6 (Separability via CEs):* Built in: W_c depends on (f, M) only through the distribution of $\mathrm{CE}(f,\mu)$ and r.

- *Axiom 7 (Additive aggregation):* Built in via the expectation $\mathbb{E}_{\mu\sim M}[\cdot]$.
- *Axiom 8 (Loss-averse kink):* Holds by construction of ϕ_r from (v_+, v_-, λ) with the stated shape and slope ratio.
- *Axiom 9 (Normalization):* u is unique up to positive affine transformations, and ϕ_r up to a positive factor that can be absorbed by u (Lemma 8).
- *Axiom 10 (Continuity in M):* If $M_n \Rightarrow M$ and $\mathrm{CE}(f,\cdot)$ is bounded and continuous on a compact set, then $\phi_r \circ (\mathrm{CE}(f,\cdot) - r)$ is bounded and continuous, so by Portmanteau, $\mathbb{E}_{M_n}[\phi_r(\mathrm{CE}(f,\mu) - r)] \to \mathbb{E}_M[\phi_r(\mathrm{CE}(f,\mu) - r)]$; hence $W_c(f, M_n) \to W_c(f, M)$.

Q.E.D.

Proof of Proposition 1

Proof. Symmetry and the definition $\phi_r(d) = v_+(d)\mathbf{1}\{d \geq 0\} - \lambda v_-(-d)\mathbf{1}\{d < 0\}$ give

$$\mathbb{E}[\phi_r(D)] = \tfrac{1}{2}\,\mathbb{E}[\phi_r(D) + \phi_r(-D)] = \tfrac{1}{2}\,\mathbb{E}[\,v_+(|D|) - \lambda v_-(|D|)\,]\,.$$

By (7), for every $t > 0$ we have $v_+(t) - \lambda v_-(t) \leq 0$, hence the integrand is a.s. ≤ 0 and strictly < 0 on any event where $|D| > 0$ and the inequality is strict. Therefore $\mathbb{E}[\phi_r(D)] \leq 0$ (strict < 0 under the stated condition), so $W_c(f, M) = u(r) + \mathbb{E}[\phi_r(D)] \leq u(r) = W_c(g, \delta)$, with strict inequality as claimed.

For the special case $v_- = v_+$, we have $v_+(t) - \lambda v_+(t) = (1-\lambda)v_+(t) < 0$ for all $t > 0$, which yields strict aversion whenever $\mathrm{P}(|D| > 0) > 0$. □

Proof of Proposition 2

Proof. We prove (a)–(b) and then apply the intermediate value theorem.

Step 1: Continuity of ΔW. Fix any sequence $p_n \to p$. Because u is continuous and strictly increasing, $r(p_n) \to r(p)$. For each ω, the map

$$(d, r) \mapsto \phi_r(d) \quad = \quad v_+(d)\mathbf{1}\{d \geq 0\} - \lambda v_-(-d)\mathbf{1}\{d < 0\}$$

is continuous in (d, r) on the compact domain $d \in [x_L - r, x_H - r]$, $r \in [x_L, x_H]$ (since $v_\pm$ are continuous and $v_\pm(0) = 0$). Hence $\phi_{r(p_n)}(X - r(p_n)) \to \phi_{r(p)}(X - r(p))$ pointwise almost surely.

Moreover, $\big|\phi_r(\,X - r\,)\big|$ is bounded uniformly over $r \in [x_L, x_H]$, because $X \in [x_L, x_H]$ implies $|X - r| \leq |x_H - x_L|$ and $v_\pm$ are continuous on a compact interval, hence bounded. Dominated convergence then yields

$$\Delta W(p_n) = \mathbb{E}\big[\phi_{r(p_n)}(X - r(p_n))\big] \xrightarrow[n\to\infty]{} \mathbb{E}\big[\phi_{r(p)}(X - r(p))\big] = \Delta W(p),$$

establishing continuity.

Step 2: Strict monotonicity in p. Let $p_1 < p_2$; then $r_1 := r(p_1) < r_2 := r(p_2)$. Fix ω and set $x := X(\omega)$. Since $r \mapsto x - r$ is strictly decreasing and $\phi_r(\cdot)$ is (by construction) the *same* strictly increasing function of its scalar argument $d = x - r$ for every fixed reference level,[9] we have the pointwise strict inequality

$$\phi_{r_2}(x - r_2) \;=\; \phi(x - r_2) \;<\; \phi(x - r_1) \;=\; \phi_{r_1}(x - r_1).$$

Taking expectations gives $\Delta W(p_2) < \Delta W(p_1)$. Therefore ΔW is strictly decreasing in p.

Step 3: Signs at the endpoints. Because $r(0) = \mathrm{CE}(g_0) = x_L$, we have

$$\Delta W(0) = \mathbb{E}[\phi_{x_L}(X - x_L)] = \mathbb{E}\big[v_+\big(X - x_L\big)\big] \quad \text{(since } X \geq x_L \text{ a.s.).}$$

If $\mathrm{P}(X > x_L) > 0$, then $v_+(X - x_L) > 0$ on a set of positive probability (as v_+ is strictly increasing and $v_+(0) = 0$), hence $\Delta W(0) > 0$.

Similarly, $r(1) = \mathrm{CE}(g_1) = x_H$, so

$$\Delta W(1) = \mathbb{E}[\phi_{x_H}(X - x_H)] = \mathbb{E}\big[-\lambda\, v_-\big(x_H - X\big)\big] \quad \text{(since } X \leq x_H \text{ a.s.),}$$

and if $\mathrm{P}(X < x_H) > 0$ then $v_-(x_H - X) > 0$ with positive probability, so $\Delta W(1) < 0$.

Step 4: Existence and uniqueness of $\bar{p}$. By Step 1, ΔW is continuous on the compact interval $[0, 1]$; by Step 3, $\Delta W(0) > 0$ and $\Delta W(1) < 0$. The intermediate value theorem then yields at least one $\bar{p} \in (0, 1)$ with $\Delta W(\bar{p}) = 0$. Step 2 (strict monotonicity) implies this zero is unique, and the sign pattern follows: $\Delta W(p) > 0$ for $p < \bar{p}$ and $\Delta W(p) < 0$ for $p > \bar{p}$.

Nondegeneracy caveat. If (f, M) is degenerate at a single certainty equivalent $X \equiv x^\star$, then $\Delta W(p) = \phi\big(x^\star - r(p)\big)$ is still continuous and strictly decreasing in p; the above argument goes through unchanged, except that one of the endpoint inequalities may be weak if $x^\star$ equals the corresponding endpoint. □

Remarks.

1. The proof does *not* require symmetry of M nor concavity/convexity of $v_\pm$; it uses only: (i) the representation of Theorem 1; (ii) continuity and strict monotonicity of $r(\cdot)$; and (iii) strict monotonicity of $v_\pm$ (hence of ϕ) with $v_\pm(0) = 0$, $\lambda > 0$. The stronger shape restrictions in Axiom 8 ensure the comparative statics of Proposition 1 but are not needed here.

2. If, in addition, the environment is *outcome-symmetric* in the sense that the map $p \mapsto r(p)$ is midpoint-symmetric around $p = \frac{1}{2}$ in *levels* (e.g., u is affine so $r(p) = p\, x_H + (1-p)\, x_L$), and if the law of X is symmetric around $r(\frac{1}{2})$, then one obtains the mirror identity $\Delta W(1-p) = -\Delta W(p)$,

[9] Formally, for any r we evaluate the same piecewise function $\phi(d)$ at $d = x - r$, with the sign cutoff at $d = 0$ (i.e., at $x = r$). Because $v_\pm$ are strictly increasing on $[0, \infty)$ and $\lambda > 0$, the function $d \mapsto \phi(d)$ is strictly increasing on $\mathbb{R}$.

hence $\bar{p} \geq \frac{1}{2}$ and the "seeking for small p, aversion for large p" regions are exact mirrors around $\frac{1}{2}$.

Proof of Proposition 3

Proof. Under (c), the index is

$$\phi_r(d) = \begin{cases} v(d), & d \geq 0, \\ -\lambda\, v(-d), & d < 0. \end{cases}$$

By symmetry (a), for any integrable function $\psi(|d|)$,

$$\mathbb{E}\big[\psi(|D|)\,\mathbf{1}\{D \geq 0\}\big] = \mathbb{E}\big[\psi(|D|)\,\mathbf{1}\{D < 0\}\big] = \tfrac{1}{2}\,\mathbb{E}[\psi(|D|)].$$

Hence

$$\mathbb{E}[\phi_r(D)] = \mathbb{E}\big[v(|D|)\,\mathbf{1}\{D \geq 0\}\big] \;-\; \lambda\,\mathbb{E}\big[v(|D|)\,\mathbf{1}\{D < 0\}\big] = \tfrac{1}{2}\,(1-\lambda)\,\mathbb{E}\big[v(|D|)\big].$$

The same identity holds for D':

$$\mathbb{E}[\phi_r(D')] = \tfrac{1}{2}\,(1-\lambda)\,\mathbb{E}\big[v(|D'|)\big].$$

Because v is (strictly) increasing and $|D'| \leq_{\text{FOSD}} |D|$, we have

$$\mathbb{E}\big[v(|D'|)\big] \;\leq\; \mathbb{E}\big[v(|D|)\big],$$

with strict inequality if $\mathrm{P}(|D'| < |D|) > 0$ and v is strictly increasing on a set of positive mass. Multiplying by the negative constant $\frac{1}{2}\,(1-\lambda)$ yields

$$\mathbb{E}[\phi_r(D')] \;\geq\; \mathbb{E}[\phi_r(D)],$$

with strict inequality under the stated condition. Adding $u(r)$ to both sides completes the proof. $\square$

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