Can Loss Aversion Explain Ambiguity Aversion?

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Abstract

I propose a new model of preferences over uncertain outcomes to explain ambiguity aversion. The model is presented in the context of the classic Ellsburg two-urn problem, and simple numerical examples are provided to demonstrate the model's ability to capture observed behavior. The model combines the insights of loss aversion (Kahneman and Tversky, 1979) with the two-stage approach of previous models, primarily that of Segal (1987). The model is similar in flavor to the *vector expected utility* model proposed by Siniscalchi (2009), in which acts are evaluated with respect to their expected utility combined with an adjustment function. More generally, Grant and Polak (2013) show that the model of Siniscalchi (2009)—and several other models of ambiguity aversion—are special cases of what they call *mean-dispersion preferences*, in which acts are evaluated with respect to their mean utility accounting for deviations from the mean. The key difference in the model I propose is that acts are evaluated differently depending on whether they are presented in isolation or alongside another act (capturing the insights of Fox and Tversky (1995)), and preferences are centered around a *reference utility* rather than the expected utility. Because of this, the model I propose is not a form of mean-dispersion preferences (Grant and Polak, 2013) or even the more general dispersion aversion model of Chambers et al. (2014). The examples provided demonstrate the model's ability to capture observed behavior across several different settings, and I believe the model provides an intuitive explanation for ambiguity aversion.

What is presented here is a general outline of the *idea* of the model. It is of course very preliminary, and the purpose of it is simply to convey the conceptual approach that I am interested in taking. In future work on this subject, my hope is to further develop the model, as well as to design some experiments that can test its performance and potentially contribute a deeper understanding of how individuals make decisions under uncertainty.

1 Introduction

A continuing open question in economics is why individuals display aversion to ambiguity. This phenomenon was first introduced by Ellsberg (1961) in thought experiments which have since become known as *Ellsberg Urns*. In this paper, I propose a new model of decision making that incorporates loss aversion to explain the existence of ambiguity aversion. To motivate the model, I focus my discussion on the two-color urn thought experiment first introduced by Ellsberg.

The two-color urn experiment can be set up as follows: there are two urns, each containing 100 balls. In the first urn, there are 50 red and 50 black balls. In the second urn, the mixture of red and black balls is ambiguous. The decision maker (DM) can choose to place a bet on the first urn or the ambiguous urn. Whichever urn the DM chooses, they can select to place their bet on either a red ball or a black ball. One ball is then randomly drawn from the selected urn, and if the color matches the DM's bet, the DM is awarded some payoff (say \$100). If the color does not match the DM's bet, the DM receives nothing (that is, \$0). Regardless of which urn the DM chooses to place their bet on, their odds of winning are 1/2. However, despite this fact, in practice it has been consistently observed that DMs prefer the first urn over the second, ambiguous urn. That is, individuals display ambiguity aversion.

Standard expected utility theory predicts that bets over the two urns should be equivalent, and therefore cannot explain the observed behavior. In response, many alternative theories of ambiguity have been developed in an attempt to explain what has been coined *Ellsberg preferences*. Among these models, at least three distinct approaches have been used. The first approach, the *Maxmin Expected Utility* model developed by Gilboa and Schmeidler (1989), assumes DMs cannot form a prior for the ambiguous urn and instead consider a set of possible priors, evaluating the bet based on the minimum expected utility obtained over all priors in the set. In our current example, if the DM considers all distributions of red and black balls to be possible, their expected utility will be minimized when the ambiguous urn happens to contain 0 red balls and 100 black balls (assuming the DM bets on red). As such, under the *Maxmin Expected Utility* model the DM will prefer the first urn (with a guaranteed 50% chance of winning) to the ambiguous urn. A second approach has been to allow DMs to have beliefs about the colors of the balls in the ambiguous urn that are represented by non-additive probability measures (Schmeidler, 1989). In our example, the DM may have beliefs represented by a *capacity* $\nu(\cdot)$, where

$$\nu(s_r, s_b) = 1, \text{ and } \nu(s_r) = \nu(s_b) = \varepsilon, \tag{1}$$

for some $\varepsilon \in [0, 1/2)$. Expected utility is then evaluated using the *Choquet Integral*. A third approach has been to model ambiguity as part of a “two-stage” model, separating the first-stage horse lottery from the roulette lottery realized in the second stage (Segal, 1987; Klibanoff et al., 2005). It is within this approach that I develop my new model.

Unlike previous models, I develop a new model which incorporates the insights of loss aversion (Kahneman and Tversky, 1979) into individuals' evaluation of the first-stage uncertainty. That is, when comparing the first urn with the ambiguous urn, the DM forms priors about the possible distributions of colors present in the ambiguous urn, and they evaluate these distributions relative to a *reference distribution*. While the reference distribution used could potentially vary across individuals and settings, in our current example we have a straightforward choice: the DM should compare all possible distributions to the 50-50 distribution they receive by choosing the first urn. Any distribution that provides worse odds than the 50-50 bet is considered to be a loss, and any distribution providing better odds is considered a gain. Because individuals are assumed to be loss averse, they will avoid the ambiguous urn because the potential losses outweigh the potential gains.

This model has a significant advantage over previous two-stage models that model ambiguity aversion as concavity of preferences over first-stage outcomes, such as the *Smooth Ambiguity Preferences Model* of Klibanoff et al. (2005). These models are unable to explain why individuals display ambiguity aversion in some cases and ambiguity preference in others, depending on the relative probabilities and whether payoffs are framed in terms of gains or losses. For a given individual, such models must either always predict ambiguity aversion or always predict ambiguity preference. By allowing for loss aversion over first-stage uncertainty, the model I propose can explain such reversals, just as prospect theory has been able to do for risk aversion.

Finally, this new model is also able to provide an intuitive explanation for the recent finding that individuals prefer larger ambiguous urns over smaller ones (Filiz-Ozbay et al., 2022). As the size of the ambiguous urn increases, the set of possible distributions becomes less extreme.[1] As such, a DM who displays loss aversion over first-stage uncertainty will prefer the larger urn.[2]

In what follows, I provide a rough outline of the model. While the model will hopefully be as general as possible, I center the following discussion around the two-urn setting previously introduced.

[1]An urn with only two balls has only three possible distributions: $(2r, 0b)$, $(1r, 1b)$, or $(0r, 2b)$. For an urn with $N = 100$ balls, there are $N + 1 = 101$ total possible distributions, but the extremes remain the same as for the $N = 2$ urn. That is, there are now many more (98 more, to be exact) possible distributions which are not as extreme as those available in the smaller urn.

[2]This is assuming the DM has beliefs about the measure $\mu(\cdot)$ which are centered around the reference distribution $(50r, 50b)$ and their uncertainty over the measure $\mu(\cdot)$, represented by the measure $M(\cdot)$ over the space $\Delta(S)$, assigns nonzero probability to all $\mu \in \Delta(S)$.

2 Theory

I model the ambiguity as a two-stage process. The general setup follows that of Segal (1987). In the first stage, a probability distribution μ over the state space S is realized, being randomly drawn with probability measure M over $\Delta(S)$. In the first urn, the first stage is degenerate, so that the probability measure M collapses to $\mu = (50r, 50b)$ with probability 1. For the second urn, all combinations of red and black balls are possible, and the DM must form beliefs about the probability measure M over all distributions $\mu \in \Delta(S)$. In the first stage, a probability distribution $\mu \in \Delta(S)$ is realized. In the second stage, a state s is then drawn based on the probability distribution μ. As done in previous models, I assume the DM views the two stages as separate and distinct. I also assume that all measures have finite support.

The DM views the basic act $f(\cdot) = (...; x_j, E_j; ...)$ as occurring within the two-stage prospect $(...; (...; x_j, E_j; ...), \mu_k; ...)$, where μ_k is the probability distribution over states that is realized in the first stage, and E_j is the event realized under distribution μ_k in the second stage. Given an act $f(\cdot) = (...; x_j, E_j; ...)$, each probability distribution $\mu \in \Delta(S)$ induces a simple lottery of the form $(...; x_j, \mu(E_j); ...)$. Thus, each basic act $f(\cdot)$ can be represented as a two-stage objective lottery of the form $(...; (...; x_j, \mu_k(E_j); ...), M(\mu_k); ...)$, where $\mu_k(E_j)$ is the probability of event E_j occurring given the probability distribution μ_k, and $M(\mu_k)$ is the individual's *personalistic* or *subjective probability* of distribution μ_k being realized in the first stage.

Following Segal (1987), I assume the individual has a preference function $V(\cdot)$ defined over single-stage lotteries, and they are able to use this preference function to determine the *certainty equivalent* of each single-stage lottery. Thus, the DM uses $V(\cdot)$ to determine the certainty equivalent of each single-stage lottery induced by each probability distribution $\mu \in \Delta(S)$. That is, for each probability distribution $\mu \in \Delta(S)$, the DM calculates the certainty equivalent $CE(f, \mu)$ for the lottery induced by μ, such that

$$V(CE(f, \mu), 1) = V(...; x_j, \mu(E_j); ...).$$

The two-stage lottery faced by the DM can now be rewritten as

$$(...; (...; x_j, \mu_k(E_j); ...), M(\mu_k); ...) = (...; CE(f, \mu_k), M(\mu_k); ...). \tag{2}$$

If the individual evaluates this lottery again using the function $V(\cdot)$, this yields the preference function

$$W(f(\cdot)) \equiv V(...; CE(f,\mu_k), M(\mu_k); ...).$$

In this case, if the preference function $V(\cdot)$ over objective lotteries is expected utility, the final preference function $W(f(\cdot))$ just becomes subjective expected utility, which is unable to explain Ellsberg preferences. Thus, rather than using expected utility, Segal (1987) uses the rank-dependent expected utility form of Quiggin (1982) for $V(\cdot)$.

From here I take a different approach. I use the expected utility form for $V(\cdot)$,[3] and I assume that individuals form additive probability measures M over the possible probability distributions $\mu \in \Delta(S)$. However, I assume that individuals do not have the same preference function for subjective (first-stage) and objective (second-stage) lotteries. In particular, subjects have the expected utility preference function $V(\cdot)$ for second-stage lotteries, but have a different preference function $\psi(\cdot)$ for first-stage lotteries. The preference function $\psi(\cdot)$ over first-stage lotteries incorporates loss aversion, and is evaluated with respect to some reference lottery $\mathbf{P}_r$. For a lottery $\mathbf{P} = (...; \alpha_i, p_i; ...)$ with $\alpha_1 > \cdots > \alpha_m > \cdots > \alpha_n$, where $\alpha_m = \inf\{\alpha : \ \alpha \geq r, \ r \equiv U^{-1}(V(\mathbf{P}_r))\}$, the preference function takes the form

$$\psi(\mathbf{P}, r) = U(r) + \sum_{i=1}^{m} \upsilon_+(\alpha_i - r) \cdot p_i + \sum_{j=m+1}^{n} \upsilon_-(\alpha_j - r) \cdot p_j, \tag{3}$$

where

$$U(x) = \begin{cases} \upsilon_+(x) & \text{for } x \geq 0 \\ \upsilon_-(x) & \text{for } x < 0 \end{cases} \tag{4}$$

and, consistent with loss aversion, $\upsilon_+(x) > 0$, $\upsilon_-(x) < 0$, $\upsilon_+'(x) < \upsilon_-'(-x)$, $\upsilon_+''(x) \leq 0$, and $\upsilon_-''(-x) \geq 0$ for all $x > 0$; and $\upsilon_+(0) = \upsilon_-(0) = 0$.

In general, an exact formula for identifying the individual's reference point may be difficult to pin down. However, with respect to the motivating example of the two-urns problem, there exists a very straightforward reference point: the probability distribution $\mu = (50r, 50b)$ available to the DM with certainty via the first urn. Denoting this probability distribution as μ_r, the individual compares the certainty equivalent of the lottery induced by each $\mu \in \Delta(S)$ with the certainty equivalent of the lottery induced by μ_r. Labeling the induced lotteries $\mathbf{P}_k = (...; x_j, \mu_k(E_j); ...)$ so that $V(\mathbf{P}_1) \geq \cdots \geq V(\mathbf{P}_m) \geq \cdots \geq V(\mathbf{P}_n)$, where $V(\mathbf{P}_m) = \inf\{V(\mathbf{P}_i) : \ U^{-1}(V(\mathbf{P}_i)) \geq U^{-1}(V(\mathbf{P}_r))\}$, the individual's preference function for

[3] That is, for a lottery $\mathbf{P} = (...; \alpha_i, p_i; ...)$, $V(\mathbf{P}) = \sum_i U(\alpha_i) \cdot p_i$.

the second urn can be written as

$$W(f(\cdot)) = V(\mathbf{P}_r) + \sum_{i=1}^{m} \upsilon_+\big(CE(f,\mu_i) - CE(f,\mu_r)\big) \cdot M(\mu_i)$$
$$+ \sum_{j=m+1}^{n} \upsilon_-\big(CE(f,\mu_j) - CE(f,\mu_r)\big) \cdot M(\mu_j). \quad (5)$$

The first term comes from observing that the certainty equivalent of a lottery $\mathbf{P}$ is given by $U^{-1}(V(\mathbf{P}))$. Without any ambiguity (i.e., without any first-stage uncertainty over the distribution μ), the above simplifies to become $W(f(\cdot)) = V(\mathbf{P}_r)$. Therefore, one restriction we will need to enforce on the formulation of the reference point will be that, for any degenerate first-stage lottery, the reference point used will always be equal to the certainty equivalent of the second-stage lottery which is realized with certainty. That is, if an individual faces a single-stage lottery $\mathbf{P}^*$, their valuation of the degenerate two-stage lottery $\{\mathbf{P}^*, 1\}$ must evaluate to $W(f(\cdot)) = V(\mathbf{P}^*)$.

We could further complicate the model by allowing $M(\mu)$ to be a non-additive measure, allowing individuals' subjective probabilities to overweight low probability events and underweight high probability events. We could then use the Choquet integral to evaluate the preference function. However, I believe this is unnecessary to capture observed behavior, and it also lacks any intuitive reasoning. Given that $M(\cdot)$ is a *subjective* probability, what reason is there to incorporate systematic over- and under-weighting of the 'true' subjective probabilities? However, this is a possibility that should perhaps be considered.

The general idea of this model is that, when comparing the ambiguous urn to the unambiguous urn, the DM considers all of the possible distributions $\mu \in \Delta(S)$ that could occur in the ambiguous urn, and they weight the 'worse' distributions more than the 'better' distributions. The worse distributions are those that provide a higher certainty equivalent. In the two urn example, the worse distributions are simply those that provide a lower probability of winning. For example, if one is planning to bet on a red ball being drawn, a worse distribution would be one such as $(40r, 60b)$, $(30r, 70b)$, etc. Because all of the possible distributions are centered around the $(50r, 50b)$ distribution of the unambiguous urn, there are an equal number of worse and better distributions possible in the ambiguous urn. As long as the individual forms $M(\cdot)$ to be symmetric around $\mu = (50r, 50b)$, each 'worse' distribution will have a mirror-image 'better' distribution that occurs with equal probability. However, because individuals are loss-averse, the worse distributions are more influential than the better distributions, causing the ambiguous urn to be considered worse than the unambiguous urn.

To help clarify how the model captures ambiguity aversion and demonstrate that the model is capable of explaining observed behavior, I now provide a few examples.

3 Examples

3.1 Preferring the unambiguous urn

Here I provide an example illustrating how the model can accommodate Ellsberg preferences in the two-urn example. Suppose the DM must choose between two gambles: (1) correctly guessing the color ball to be drawn from an unambiguous urn with 1 red and 1 black ball, or (2) correctly guessing the color ball to be drawn from an ambiguous urn with 2 total balls, each of which is either red or black. Whichever gamble the DM chooses, if they correctly guess the color to be drawn they are awarded \$100. Otherwise, they are awarded \$0. Normalize utility so that $U(\$100) = 1$ and $U(\$0) = 0$.

For the individual's subjective beliefs $M(\cdot)$, I believe it is fair to assume symmetry of beliefs between the two colors. That is, if we restrict our attention to their beliefs over only one of the colors, say red, $M(\cdot)$ must be symmetric around $\mu_m = (1r, 1b)$.[4] There are two distributions for $M(\cdot)$ that I think are reasonable for individuals to hold. The first is the uniform distribution, which in general is given by $\mu_i = 1/(N + 1)$, where N is the size of the urn (i.e., the number of balls in the urn). Since each possible distribution μ_i is uniquely identified by the number of red balls, we can think of μ_i as referring to the distribution with i red balls (and $N - i$ black balls, so $\mu_i = (i * r, (N - i) * b)$). The other reasonable distribution is the binomial distribution, given by $\mu_i = \binom{N}{i}\lambda^i(1 - \lambda)^{N-i}$, where λ is the individual's perceived probability of a red ball being placed in the urn. In our current example, $\lambda = 1/2$. In this case, the individual believes there is an underlying random process by which balls are selected to fill the ambiguous urn. Both red and black balls have a 1/2 chance of being selected to fill the urn, so that the distribution is centered around $\mu_m = (1r, 1b)$, but the randomness in the selection process creates uncertainty around the final realized distribution. To the individual, it's as if the ambiguous urn was filled by randomly selecting balls (with replacement) from an arbitrarily large urn containing λN red balls and $(1 - \lambda)N$ black balls.

To keep things simple, for the current example I will assume the individual holds uniform beliefs over the possible distributions for the ambiguous urn. Since $N = 2$, they assign

[4] For example, if the individual believes there is a 10% chance that the realized distribution will have 0 red balls (and therefore 2 black balls), they must also believe there is a 10% chance that the realized distribution will have 0 *black* balls (and 2 red balls). I will discuss relaxing this assumption in the future. However, given that in this example the first urn provides the distribution $(1r, 1b)$, I believe it is fair to assume that beliefs should be symmetric around this distribution.

probability $1/(N+1) = 1/3$ to each of the possible distributions

$$\mu_0 = (0r, 2b),\ \mu_1 = (1r, 1b),\ \text{and}\ \mu_2 = (2r, 0b). \tag{6}$$

For a bet on red, the distribution μ_i provides probability i/N of winning \$100.

Define utility to be given by $\upsilon_+(x) = (x/100)^\beta$ for $x \geq 0$ and $\upsilon_-(x) = -\lambda(-x/100)^\beta$ for $x < 0$. Following the median estimated values of β and λ reported in Tversky and Kahneman (1992), set $\beta = 0.88$ and $\lambda = 2.25$. A bet on the first urn, denoted a_1, induces the degenerate two-stage lottery $\{\mathbf{P}_1, 1\}$, so that $W(a_1) = V(\mathbf{P}_1)$. The expected utility of the first urn is given by $V(\mathbf{P}_r) = 1/2$, where I denote the lottery induced by the first urn as $\mathbf{P}_r$ since it is the reference lottery used in the evaluation of the second urn. So $W(a_1) = 1/2$.

To evaluate the second urn, we must calculate the certainty equivalent for each possible second-stage lottery. Let a_2 be a bet on red in the second urn. Then for μ_0, which is a degenerate lottery with guaranteed payoff of \$0, we have $CE(a_2, \mu_0) = 0$. For μ_2, a degenerate lottery with guaranteed payoff of \$100, we have $CE(a_2, \mu_2) = 100$. Finally, for the reference lottery μ_1, we have $CE(a_2, \mu_1) = 45.50$. Then the preference function for a bet on red for the second urn is given by

$$\begin{aligned}
W(a_2) = 1/2 + \left(\frac{100 - 45.50}{100}\right)^{0.88} \cdot (1/3) + \left(\frac{45.50 - 45.50}{100}\right)^{0.88} \cdot (1/3) \\
- 2.25 \cdot \left[\left(-\frac{0 - 45.50}{100}\right)^{0.88}\right] \cdot (1/3) \\
= 0.5 + 0.195 - 0.375 \\
= 0.32
\end{aligned}$$

Thus, we have that

$$W(a_1) = 0.5 > 0.32 = W(a_2)$$

and the individual prefers the bet on the unambiguous urn over the bet on the ambiguous urn.

A note on the relationship between ambiguity aversion and the concavity of the utility function

In the example just discussed, the underlying utility function is assumed to be concave over positive monetary gains. To the extent that one believes concavity of utility represents risk aversion, the previous exercise provides an example of a setting in which a risk averse individual also displays ambiguity aversion. As can easily be shown, an individual with a linear utility function—who would therefore be risk neutral—will also display ambiguity aversion as long as they experience loss aversion (i.e., $\lambda > 1$ in the previous example).[5] One might therefore assume that the amount of ambiguity aversion an individual displays will be increasing in the concavity of their utility function. However, this is the opposite of the actual relationship. As the utility function becomes increasingly concave, the individual becomes *less* averse to ambiguity (and may actually *prefer* ambiguity, depending on the size of λ). This can be most easily seen by considering an example.

Let's keep the same parameterization of the previous example, except that we will now set $\lambda = 1.5$ and $\beta = 0.1$.[6] That is, we are decreasing the amount of loss aversion and increasing the amount of risk aversion. In this case we get the following calculation:

$$
\begin{aligned}
W(a_2) &= 1/2 + \left(\frac{100-0.10}{100}\right)^{0.1} \cdot (1/3) - 2.25 \cdot \left[\left(-\frac{0-0.10}{100}\right)^{0.1}\right] \cdot (1/3) \\
&= 0.5 + 0.33 - 1.5(0.167) \\
&= 0.58
\end{aligned}
$$

Thus, we now find that $W(a_2) = 0.58 > 0.5 = W(a_1)$, so that the individual prefers the bet on the ambiguous urn over the bet on the unambiguous urn. That is, the individual now displays a preference for ambiguity.

If we reset $\lambda = 2.25$ so that the individual is as loss averse as they were before, then the above calculation becomes $W(a_2) = 0.46 < 0.5 = W(a_1)$, and we again observe that the individual is ambiguity averse, albeit less so than in the original example. This demonstrates

[5]It is also assumed that the individual has subjective beliefs over the distribution of $\mu \in \Delta(S)$ that are symmetric, and that the subjective distribution $M(\mu)$ is not degenerate (i.e., the reference distribution μ_m does not occur with probability 1). In the previous example, where the DM believes the distributions $(0r, 2b)$ and $(2r, 0b)$ both occur with probability 1/3, it is clear that they will prefer the unambiguous urn: although the 'better' distribution $(2r, 0b)$ is considered to be better than the reference distribution $(1r, 1b)$ by the exact same amount that the 'worse' distribution $(0r, 2b)$ is considered to be worse than the reference distribution, because $\lambda > 1$ (i.e., the individual experiences loss aversion) the prospect of the 'worse' distribution weighs more heavily on the individual's decision-making process and results in a preference for the unambiguous urn.

[6]This will make utility more concave over the positive domain and more convex over the negative domain, since we are assuming loss aversion.

the importance of loss aversion in observing ambiguity aversion. While higher levels of risk aversion will make an individual more likely to prefer ambiguity, loss aversion counteracts this effect.

3.2 Situation-dependent ambiguity aversion

One finding in the literature on ambiguity aversion that has been difficult to reconcile with theoretical models of decision making is the finding that individuals will display an aversion to ambiguity in some settings, while in other settings they display a preference for ambiguity. When the unambiguous odds of winning a prize are relatively high, decision-makers tend to avoid ambiguity, presumably fearing that the odds of winning will be lower for the ambiguous gamble. However, when the unambiguous odds of winning are relatively low, DMs tend to prefer ambiguity, presumably hoping that the odds will be better for the ambiguous gamble.

An example of the type of setting in which this behavior would be observed was provided by Ellsberg (2015) in what he called his *n-Color Example*. In this case, there are again two urns, each containing 100 balls. However, there are now 10 different colors. The first urn has 10 balls of each color, while the second (ambiguous) urn could contain any mixture of the 10 colors in any proportions. First consider a DM who must decide between two gambles: a_1 pays \$100 if a red ball is *not* drawn from the first (unambiguous) urn, and otherwise pays \$0; a_2 pays \$100 if a red ball is *not* drawn from the second (ambiguous) urn, and otherwise pays \$0. The bet on the unambiguous urn (a_1) pays \$100 with a known probability of 90%, while the bet on the ambiguous urn (a_2) is ambiguous. In this case, most DMs will prefer gamble a_1 over a_2. However, now consider a DM who must decide between the following two gambles: a_3 pays \$100 if a red ball is drawn from the first (unambiguous) urn, and otherwise pays \$0; a_4 pays \$100 if a red ball is drawn from the second (ambiguous) urn, and otherwise pays \$0. The bet on the unambiguous urn (a_3) pays \$100 with a known probability of 10%, while the bet on the ambiguous urn (a_4) is ambiguous. In this case, most DMs will prefer the ambiguous gamble a_4 over a_3. That is, once the unambiguous odds of winning become low enough, DMs reverse their preference for ambiguity.

This pattern of behavior is easily explained by the loss aversion model I have developed. For the comparison of a_1 and a_2, where the unambiguous probability of winning \$100 is 90%, there are many more possible distributions which provide a lower probability of winning than there are that provide a higher probability. When comparing the two gambles, there is little potential upside to picking the ambiguous gamble a_2, but there is a lot of potential downside. Thus, it makes sense that in general DMs would prefer a_1 over a_2. However, the comparison of a_3 and a_4 exhibits the opposite characteristics. The unambiguous gamble (a_3) provides only a 10% chance of winning \$100, and there are now many more possible distributions which provide a higher probability of winning than there are distributions that provide a lower probability. There is now little downside to choosing the ambiguous gamble, with a significant amount of possible upside. In this case, DMs will be much more likely to prefer the ambiguous gamble a_4 over a_3.

To demonstrate that the model can capture preference reversals such as those of Ellsberg's *n-Color Example*, keep the same parameterization of utility as the first example. That is, set $\beta = 0.88$ and $\lambda = 2.25$. To help simplify things, reduce the total number of balls in each urn to be 10, but keep all other aspects of the example the same. That is, the unambiguous urn now has 1 ball of each color, and the ambiguous urn can contain any combination of colors.

Let us first consider the comparison of gambles a_1 and a_2. Recall that a_1 pays \$100 if a red ball is *not* drawn from the first (unambiguous) urn, and pays \$0 otherwise; and a_2 pays \$100 if a red ball is *not* drawn from the second (ambiguous) urn, and pays \$0 otherwise. As before, the individual's *reference lottery* is the unambiguous lottery, which in this case pays \$100 with 90% probability and \$0 with 10% probability. The certainty equivalent of this lottery is $CE(f, \mu_r) = 88.72$.[7] If we assume the individual considers the number number of red balls in the ambiguous urn to be determined by a uniform distribution (so that $M(\mu_i) = \frac{1}{N+1}$ for all i, where i is the number of red balls in the urn and N is the size of the urn), then the individual's evaluation of the ambiguous gamble a_2 is given by

$$W(a_2) = 0.9 + \sum_{i=0}^{m} \left(\frac{CE(f, \mu_i) - CE(f, \mu_r)}{100} \right)^{0.88} \cdot \frac{1}{N+1} - 2.25 \cdot \sum_{j=m+1}^{N} \left(-\frac{CE(f, \mu_j) - CE(f, \mu_r)}{100} \right)^{0.88} \cdot \frac{1}{N+1},$$

where $m = 1$ and $N = 10$. Relative to the unambiguous urn, there are clearly many more 'worse' possible second-stage lotteries than there are 'better' ones. So even without computing the above calculation, we should expect that $W(a_2) < W(a_1)$. Indeed, the calculation comes out to $W(a_2) = -0.11 < 0.9 = W(a_1)$, showing that the individual

[7] f is an act that pays \$100 if the event "a red ball is *not* drawn" is realized and pays \$0 if the event "a red ball is drawn" is realized. Note that the outcome of the act is based only on whether the drawn ball is red or not. Therefore, all possible distributions $\mu \in \Delta(S)$ can be expressed as $(r, N - r)$, where r is the number of red balls and N is the total number of balls in the urn. This assumes that individuals do not differentiate between distributions that contain the same number of red balls (but for which the remaining balls may consist of different mixtures of the other colors). I believe this assumption is easily defended. This assumption greatly reduces the complexity of the computation of $W(\cdot)$, because it greatly reduces the number of possible distributions that must be considered.

strongly prefers the unambiguous gamble. The fact that $W(a_2)$ is calculated to be negative should be interpreted keeping in mind that this is the individual's evaluation of the ambiguous bet *in comparison* with the unambiguous bet. That is, $W(a_2) < 0$ does not imply that the DM prefers receiving \$0 with certainty over playing gamble a_2 *in general*, but only that, when gamble a_1 is their reference lottery, they would prefer to receive nothing than to have to play gamble a_2.

Now consider the comparison of gambles a_3 and a_4. Gamble a_3 pays \$100 if a red ball is drawn from the first (unambiguous) urn, and \$0 otherwise. Gamble a_4 pays \$100 if a red ball is drawn from the second (ambiguous) urn, and \$0 otherwise. In this case, the individual's *reference lottery* is the unambiguous lottery a_3, which pays \$100 with 10% probability and \$0 with 90% probability. The certainty equivalent of this lottery is $CE(f, \mu_r) = 7.31$, and the individual's evaluation of the ambiguous gamble a_4 is given by

$$W(a_4) = 0.1 + \sum_{i=m}^{N} \left(\frac{CE(f, \mu_i) - CE(f, \mu_r)}{100} \right)^{0.88} \cdot \frac{1}{N+1}$$
$$- 2.25 \cdot \sum_{j=0}^{m-1} \left(- \frac{CE(f, \mu_j) - CE(f, \mu_r)}{100} \right)^{0.88} \cdot \frac{1}{N+1},$$

where $m = 1$ and $N = 10$. Note the change of indices used for the summations. For this gamble, having more red balls increases the chances of receiving the higher payout (and therefore results in a 'better' lottery), while having fewer red balls results in a 'worse' lottery. As is clear from the indices used, there are many more 'better' possible lotteries than there are 'worse' possible lotteries. As such, the individual should be much more likely to prefer the ambiguous gamble to the unambiguous gamble. Indeed, performing the above calculation shows $W(a_4) = 0.51 > 0.1 = W(a_3)$, confirming that the ambiguous gamble a_4 is preferred to a_3.

3.3 Preferring larger ambiguous urns

As a final example, I demonstrate that the model is also able to capture the preference for larger ambiguous urns observed in Filiz-Ozbay et al. (2022). Using an experiment in which the size of urns is varied, Filiz-Ozbay et al. (2022) observe that, when comparing two ambiguous urns of different sizes, subjects tend to prefer the bet on the larger urn.[8]

My model can explain the preference for larger ambiguous urns, but we must first make a new rule for how individuals decide on a reference lottery. Unlike the previous examples,

[8]Subjects also appear to have a small preference for larger *unambiguous* urns, most likely because they are stupid.

there is no longer any unambiguous gamble to serve as a reference lottery. While we could simply add a new rule to address this particular setting, it is clear that the model needs to have a set of rules for determining the reference lottery to be used more generally. Although this should have been addressed when the model was introduced, I will instead address it here.

Rules for Selecting the Reference Lottery:

(1) Single-stage (unambiguous) lotteries are evaluated with respect to the degenerate lottery $\mathbf{P}_r = (0, 1)$, so that $r = 0$ for all single-stage lotteries. When all payoffs in lottery $\mathbf{P}$ are non-negative (as in all of these examples), $V(\mathbf{P})$ simplifies to the standard expected utility formula.

(2) When comparing a two-stage (ambiguous) lottery to a single-stage (unambiguous) lottery, the single-stage (unambiguous) lottery is used as the reference lottery.

(3) When comparing two ambiguous lotteries, each lottery is evaluated using the *reduced lottery* derived from the comparison lottery. That is, for two ambiguous lotteries defined by the act $f(\cdot)$ and measures $M(\cdot)$ and $M'(\cdot)$, the reference lottery used to evaluate the first lottery is given by $\mathbf{P}_r = (...; f(E_j), \sum_k \mu_k(E_j) \cdot M'(\mu_k); ...)$, and vice versa.

Rules (2) and (3) really say the same thing. They both say that whenever a two-stage lottery is evaluated using $W(f(\cdot))$, the reduction of the comparison lottery is used as the reference lottery. Since the reduction of a single-stage objective lottery $\mathbf{P}$ is just $\mathbf{P}$, the objective lottery is used as the reference lottery. However, the reduction of a two-stage lottery depends on the individual's subjective beliefs about $M(\cdot)$, and therefore the reference lottery is given by the reduction of the comparison lottery.

Using these rules, we can now compare two ambiguous urns, as is done in Filiz-Ozbay et al. (2022). Consider two urns of different sizes. The first urn contains 2 balls of an unknown mixture of red and black, and the second urn contains 10 balls of an unknown mixture of red and black. Suppose the DM can bet on either a red or black ball being drawn from one of the urns, but must decide on which of the two urns they would like to place their bet. Keep the same parameterization as used in all previous examples.

In order to evaluate the first urn, of size $N = 2$, the DM must determine the reference lottery $\mathbf{P}_r$. In this setting there are only two possible events: E_1 = red is drawn, and E_2 = black is drawn. Thus, the reference lottery is given by

$$\mathbf{P}_r = \left(\$100, \sum_{k=0}^{N=10} \mu_k(E_1) \cdot M_{10}(\mu_k);\ \$0, \sum_{k=0}^{N=10} \mu_k(E_2) \cdot M_{10}(\mu_k) \right),$$

where $M_{10}(\cdot)$ is the individual's subjective beliefs regarding the second-order uncertainty in urn 2 (with $N = 10$ balls). The reference lottery will vary depending on the distribution $M_{10}(\cdot)$. However, as before, there are two straightforward distributions we can use: the uniform distribution and the binomial distribution. The uniform distribution will simplify to $\mathbf{P}_r = (\$100, 0.5;\ \$0, 0.5)$. Likewise, if the binomial distribution is assumed to be centered around 0.5 (which would make the most sense), it will simplify to the same reference lottery. The certainty equivalent of this reference lottery is given by $CE(f, \mu_r) = 45.50$, and the gamble on the smaller urn is evaluated as

$$W(a_2) = 0.5 + \left(\frac{100 - 45.50}{100}\right)^{0.88} \cdot M_2(\mu_0) + \left(\frac{45.50 - 45.50}{100}\right)^{0.88} \cdot M_2(\mu_1) - 2.25 \cdot \left[\left(-\frac{0 - 45.50}{100}\right)^{0.88}\right] \cdot M_2(\mu_2).$$

If the uniform distribution is used for $M_2(\cdot)$, we get $W_u(a_2) = 0.32$. If the binomial distribution centered around 0.5 is used, we get $W_b(a_2) = 0.37$.

To evaluate the second urn, with $N = 10$, the DM must determine the reference lottery using the expectation of the lottery for the first urn. Using either the uniform distribution or the binomial distribution centered around 0.5 for $M_2(\cdot)$ will produce the same reference lottery as before. Thus, the gamble on the larger urn is evaluated as

$$W(a_{10}) = 0.5 + \sum_{i=m}^{N} \left(\frac{CE(f, \mu_i) - CE(f, \mu_r)}{100}\right)^{0.88} \cdot M_{10}(\cdot) - 2.25 \cdot \sum_{j=0}^{m-1} \left(-\frac{CE(f, \mu_j) - CE(f, \mu_r)}{100}\right)^{0.88} \cdot M_{10}(\cdot),$$

where $m = 5$. If the uniform distribution is used for $M_{10}(\cdot)$, we get $W_u(a_{10}) = 0.33$. If the binomial distribution centered around 0.5 is used, we get $W_b(a_{10}) = 0.41$. Thus, we find that

$$W_u(a_2) = 0.32 < W_u(a_{10}) = 0.33, \text{ and}$$
$$W_b(a_2) = 0.37 < W_b(a_{10}) = 0.41.$$

That is, under both distributions the DM prefers to gamble on the larger urn. However, the effect is more pronounced using the binomial distribution.

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